PHY 711: ANALYTICAL DYNAMICS

Additional Practice Problems III

Problem 1

The velocity of a particle $\frac{dx^i}{dt}$ does not have a nice transformation property since both x^i and t change and get mixed under Lorentz transformations. For this reason, one defines a 4-velocity u^μ which transforms as a 4-vector under Lorentz transformations. The actual 3-vector v^i corresponding to the velocity can then be defined by

$$\frac{v^i}{c} = \frac{u^i}{u^0}$$

a) Consider a particlemoving aong the x_1-direction, so that $u^\mu = (u^0, u^1, 0, 0)$. Consider the velocity of the particle in a frame moving with velocity $w^i = (w^1, 0, 0)$. This can be done by carrying out a Lorentz transformation and then forming the ratio as above. In this way, obtain the relativistic law for addition of velocities along the same direction.
b) Consider the same set-up but with the frame moving along the x_2-direction, so that $w^i = (0, w^2, 0)$. What is the addition law for velocities now?

Problem 2

A particle moves in one dimension x under the influence of a potential

$$V(x) = -b\frac{1}{\cosh^2 ax}$$

where a, b are positive constants. Use the Hamilton-Jacobi method to find the solution (trajectory of the particle) for negative values of the energy. (This potential is important in the context of soliton solutions for the Korteweg-de Vries equation for surfaces waves in a shallow body of water.)

Problem 3

The Lagrangian for a spinning top on the floor (i.e., subject to gravity) is given by

$$\mathcal{L} = \frac{1}{2}I_1(\dot{\theta}^2 + \sin^2\theta\,\dot{\psi}^2) + \frac{1}{2}I_3(\dot{\varphi} + \dot{\psi}\cos\theta)^2 - Mgh\,\cos\theta$$

(This is what is called Lagrange top; we discussed this in class.)
a) Identify the canonical momenta and the Hamiltonian.
b) Write down the Hamilton-Jacobi equation and find the general solution.

Problem 4

The Lagrangian for a nonrelativistic particle interacting with an external magnetic field is given by

$$\mathcal{L} = \tfrac{1}{2}\, m\dot{x}_i\dot{x}_i + eA_i\dot{x}_i$$

where $i = 1, 2, 3$ and there is summation over the index i.

a) Obtain the Hamiltonian in terms of the canonical momenta and coordinates.

b) Now consider a uniform magnetic field in the the z direction with $A_i = -\frac{1}{2}eB\epsilon_{ij}x^j$, for $i, j = 1, 2$, $A_3 = 0$. Ignore motion in the z-direction in what follows. Define a change of variables to

$$\begin{aligned}
Q_1 &= \frac{1}{\sqrt{eB}}\,(p_1 + \tfrac{1}{2}eBx_2) \\
Q_2 &= \frac{1}{\sqrt{eB}}\,(p_2 + \tfrac{1}{2}eBx_1) \\
P_1 &= \frac{1}{\sqrt{eB}}\,(p_2 - \tfrac{1}{2}eBx_1) \\
P_2 &= \frac{1}{\sqrt{eB}}\,(p_1 - \tfrac{1}{2}eBx_2)
\end{aligned}$$

Obtain the Poisson brackets $\{Q_i, Q_j\}$, $\{P_i, P_j\}$, $\{Q_i, P_j\}$.

c) Write the Hamiltonian in terms of these new variables. Obtain the equations of motion for all Q_i, P_i, $i = 1, 2$.

d) What are the conserved quantities? Solve the equations of motion to obtain the general solution.

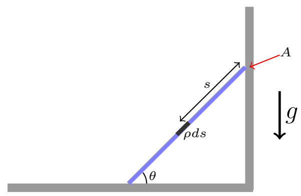
Problem 5

A charged particle which moves in a medium (with refractive index n) with a speed v which is greater than the speed of light in the medium (c/n) will emit radiation, known as the Cerenkov radiation. Consider Cerenkov radiation from an electron in such a medium; this may be considered as the emission of a photon, namely, the process $e^- \to e^- + \gamma$. The four-vector of energy and momentum for the emitted photon may be taken as $(\hbar\omega/c, \hbar\vec{k})$ with $\vec{k}\cdot\vec{k}/n^2 = \omega^2/c^2$. (Here $\hbar$ is Planck's constant.) Show that the radiation is in cone and obtain the opening angle of the cone.

Problem 6

A rod of very small thickness rests against a wall as shown. The rod can slide down, with point of contact (shown as A) always touching the wall.

a) Obtain the Lagrangian describing this motion. (Hint: *Consider a small mass element of mass ρds at a distance s from the point A, as shown. Obtain the coordinates of this element to construct the kinetic energy for this element and then integrate over the length of the rod.*)
b) Identify the canonical momentum and the Hamiltonian.
c) Write down the Hamilton-Jacobi equation and its solution as an integral over θ. (The integral can be done in terms of elliptic functions, but you do not have to do that.)



Problem 6

Problem 7

A particle of mass m is constrained to move on a circle of radius R in the (x, y)-plane, $x^2 + y^2 = R^2$. There is a potential energy for the motion given by $V = \frac{1}{2}(ax^2 + by^2)$, where a, b are positive constants with $a > b$.
a) Write down the Lagrangian and the Hamiltonian.
b) Identify the equilibrium points and determine which are the stable equilibrium points.
c) Determine the frequency of small oscillations around the stable equilibrium points.

Problem 8

In quantum mechanics, one is familiar with the step-up and step-down operators $a^\dagger$ and a. Here we will consider a classical analog. The Hamiltonian is given by

$$H = \frac{p^2}{2m} + \frac{m\omega^2}{2} q^2$$

Define the complex combinations

$$\frac{p}{\sqrt{2m\omega}} + i\sqrt{\frac{m\omega}{2}}\, q = a_{\rm cl} = \sqrt{I}\, e^{i\varphi}, \qquad \frac{p}{\sqrt{2m\omega}} - i\sqrt{\frac{m\omega}{2}}\, q = a^*_{\rm cl} = \sqrt{I}\, e^{-i\varphi}$$

a)Write down the Hamiltonian in terms of the new variables I and φ.
b) Show that the change of variables from (p, q) to (I, φ) is canonical.